

Quantum Circuits for d -level Systems

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joint with

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Outline

- I. Introduction to Qudits (Quantum Multi-level Logics)
- II. Universality & Selection Rules
- III. Complexity and Lower Bounds
- IV. Asymptotically Optimal Qudit State-Synthesis
- V. *QR*-based Asymptotically Optimal Circuits

Qudits, i.e. Quantum Multi-level Logics

- **Q.C.** replaces bit with qubit: **two state quantum system**, states $|0\rangle, |1\rangle$
- **Multi-level logic**: not bit but dit, values $0, 1, \dots, d-1$
- **Qudit**: states $|0\rangle, |1\rangle, \dots, |d-1\rangle$
 - Single qudit state space $\mathcal{H}(1, d) = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \oplus \dots \oplus \mathbb{C}|d-1\rangle \cong \mathbb{C}^d$
 - n -qudit state space

$$\mathcal{H}(n, d) = \bigotimes_1^n \mathcal{H}(1, d) = \bigoplus_{\bar{c} \in (\mathbb{Z}/d\mathbb{Z})^n} \mathbb{C}|\bar{c}\rangle \cong \mathbb{C}^{dn}$$

Emulating Qudits with Qubits

- **Scheme #1:** Pack each qudit into $\lceil \log_2 d \rceil$ qubits, $n \lceil \log_2 d \rceil$ total
 - Qubit circuits yield qudit circuits
 - Some virtual two-qubit gates are qudit-local
 - **Heuristic:** Hilbert space dimensions are fungible
- **Scheme #2:** Pack n qudits into $\lceil \log_2 d^n \rceil \neq n \lceil \log_2 d \rceil$ qubits
 - Single qudit levels spread across multiple qubits
 - Circuit diagrams do not translate: **not** that fungible!

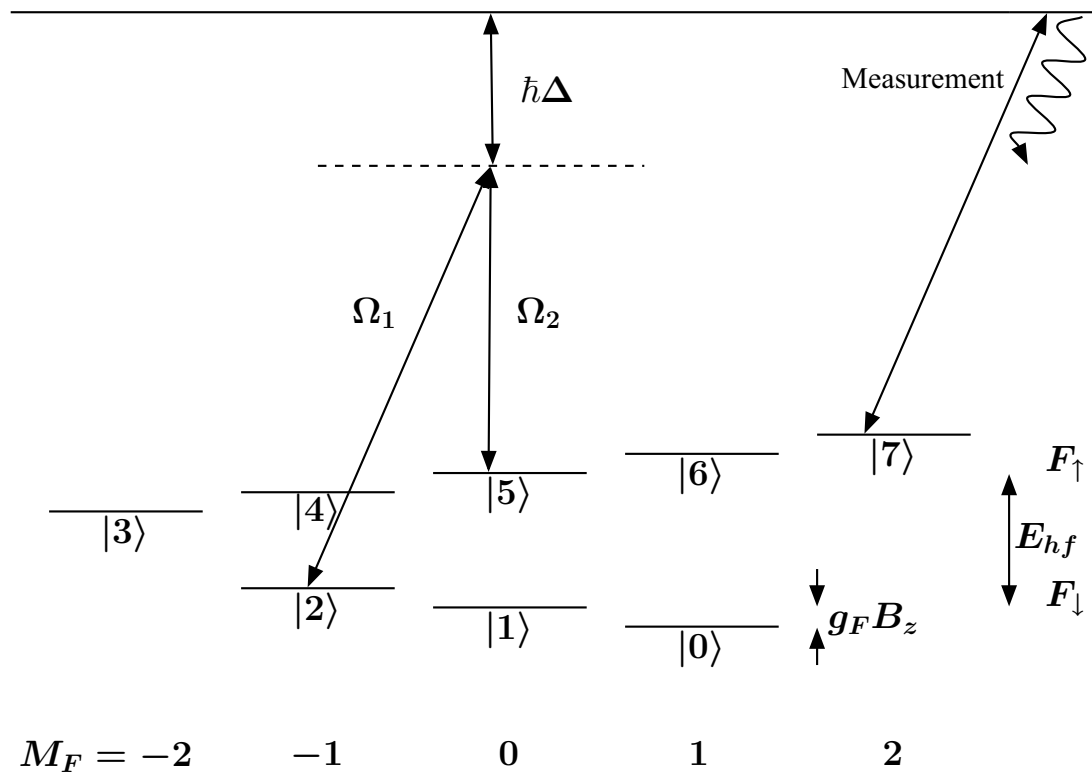
Why Qudits?

- More efficient use of physical system, **if** all states may be controlled
- **Trade-off**: If $d = 2^\ell$, fewer entangling gates, more local op's
- More natural for some computations, especially (Hoyer, q-ph/9702028)
Fourier transform of $\mathbb{Z}/d^n\mathbb{Z}$ in case $\gcd(d, 2) = 1$
- Perhaps **less decoherence** or **better error correction** if $d \neq 2$

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Motivation: Quoctet ^{87}Rb Hyperfine Levels ($d = 8$) & Allowed Couplings



Motivation: Quoctet ^{87}Rb Hyperfine Levels ($d = 8$) & Allowed Couplings Cont.

- Template One-qudit Hamiltonians:
$$\begin{aligned} H_{jk}^x &= |k\rangle\langle j| + |j\rangle\langle k| \\ H_{jk}^y &= i|k\rangle\langle j| - i|j\rangle\langle k| \end{aligned}$$

- Certain levels $|j\rangle, |k\rangle$ allow **atom-laser Hamiltonian**:

$$H_{AL,jk} = \cos(\phi)H_{jk}^x + \sin(\phi)H_{jk}^y$$

- **Selection rule**: only allow H_{jk}^x, H_{jk}^y for certain pairs (j, k) :

$$(0,5), (0,6), (0,7), (1,4), (1,6), (2,3), (2,4), (2,5)$$

Universality Problems

- Notation: $U(\ell) = \{V \in \mathbb{C}^{\ell \times \ell} ; V\bar{V}^T = I_\ell\}$
- $U(d)$ for one-qudit unitary evolution, $U(d^n)$ for n -qudits
- **Problem #1:** Can we build all one-qudit unitary evolutions using time evolution by Hamiltonians allowed by our selection rule?
- **Problem #2:** Given a nice two-qudit Hamiltonian,
e.g. $\mathbf{H} = |d-1\rangle \otimes |d-1\rangle \langle d-1| \otimes \langle d-1|$,
can we construct evolutions for all $V \in U(d^{2n})$?

Universality Technique: QR Decompositions

- **QR Decomposition:** Any $M \in \mathbb{C}^{\ell \times \ell}$ factors $M = RU$, $U\bar{U}^T = I_\ell$, R semi-upper triangular
 - Columns of U : Hermitian o.n. basis of \mathbb{C}^ℓ
 - One method: Gram Schmidt o.n. of columns of M
 - Other methods: build unitaries U_1, \dots, U_p , each adding subdiagonal 0's to partial products $U_k U_{k+1} \dots U_p M$
 - * U_q may be **Givens rotations**
 U_q act as $V \in U(2)$ on $\mathbb{C}|j\rangle \oplus \mathbb{C}|k\rangle$, identity else
 - * U_q may be **Householder reflections**
For fixed $|\psi\rangle$, reflect so $U_q|\psi\rangle = \sqrt{\langle\psi|\psi\rangle}|0\rangle$

QR Reduction Using Givens Rotations

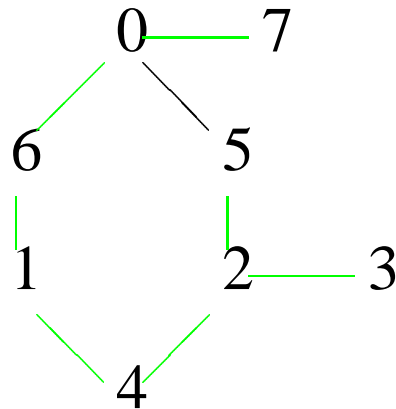
$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \xrightarrow{V_{2,3}} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \end{pmatrix} \xrightarrow{V_{1,2}}$$

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \xrightarrow{V_{2,3}} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \xrightarrow{V_{0,1}}$$

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \xrightarrow{V_{2,3} \circ V_{1,2}} \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

Universality Technique: *QR* Decompositions Cont.

- **Coupling graph**: short-hand describing allowed $H_{j,k}^x, H_{j,k}^y$



- Notation: any unitary 2×2 matrix V , with V_{jk} the associated Givens rotation in $\mathbb{C}|j\rangle \oplus \mathbb{C}|k\rangle$

Coupling Graph-Based QR Decomposition: One-qudit Universality

- **Euler angle technique:** Givens rotations for coupled $|j\rangle, |k\rangle$

$$V_{jk} = e^{i\phi} \exp(it_0 H_{jk}^x) \exp(it_1 H_{jk}^y) \exp(it_2 H_{jk}^x)$$

- **Problem #1 Restatement:** Build $U \in U(d)$ using a restricted set of Givens rotations encoded in the coupling graph
- This is possible using an **optimal number of V_{jk}** :
 - Build **spanning tree** of coupling graph
 - Introduce zero at entry of leaf, remove leaf, etc.

Two-qudit Generalizations of CNOT

- **Generalization #1:** $\Lambda_1(\sigma^x \oplus I_{d-2})$,
exchanging $|0\rangle \leftrightarrow |1\rangle$, $|\ell\rangle \mapsto |\ell\rangle$ else
- **Increment gate INC:** one qudit modular addition,
 $\text{INC}|\ell\rangle = |\ell + 1 \bmod d\rangle$; INC is denoted in circuits as \oplus
- **Generalization #2:** $\text{CINC} = \Lambda_1(\text{INC})$
CINC denoted in circuits by old CNOT symbol
- CINC constructible from $U(d)$, $\Lambda_1(\sigma^x \oplus I_{d-2})$
- Using ancillas, $U(d) \sqcup \{\Lambda_1(V)\}$ is exact-universal

QR Technique for Two-qudits, i.e. $U(d^2)$

- **Barenco et al (q-ph/9503016):** Any 2×2 unitary $V = e^{i\varphi} A \sigma^x B \sigma^x C \sigma^x$, such that $ABC = I_d$

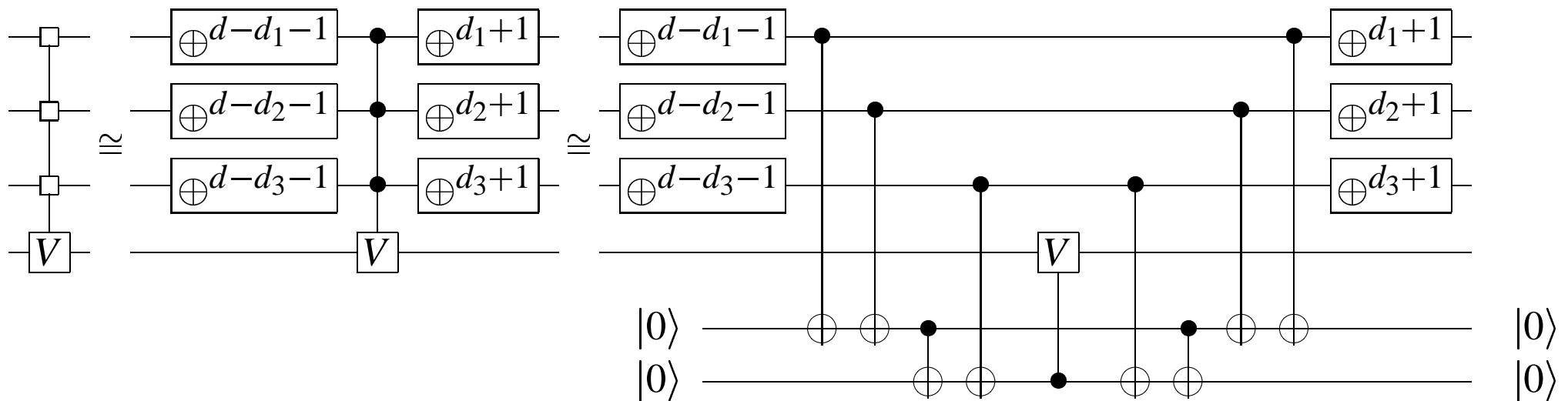
- Determinant one Givens rotation V_{jk} , implicitly buffer A, B, C by I_{d-2} :

$$\begin{aligned}
 \Lambda_1(V_{jk}) &= I_{d^2-d} \oplus (V_{jk}) \\
 &= (ABC) \oplus (ABC) \oplus \cdots \oplus (ABC) \oplus (A \sigma^x B \sigma^x C) \\
 &= (A \otimes I_d) \Lambda_1(\sigma^x \oplus I_{d-2}) (B \otimes I_d) \Lambda_1(\sigma^x \oplus I_{d-2}) (C \otimes I_d)
 \end{aligned}$$

- **Another QR Decomposition:** Gate library $U(d)$, $\Lambda_1(\sigma^x \oplus I_{d-2})$ builds any $V \in U(d^2)$

QR Technique: Two Qudit Universal $\implies n$ -Qudit Universal

- Suffices to use two-qudits to build any Givens rotation in $U(d^n)$



QR-Universality Summary Slide

- Laser Hamiltonians $H_{jk}^x, H_{jk}^y \implies$ all **one-qubit evolutions in $U(d)$**
 - Coupling graph must be connected
 - If connected, no extra pulses required in general
- One-qudit unitaries, $\bigwedge_1(\sigma^x \oplus I_{d-2}) \implies$ all **two-qudit unitaries $V \in U(d^2)$**
 - Much simpler construction for $\text{CINC} = \bigwedge_1(\text{INC})$
- Two-qudit unitaries \implies all **n -qudit unitaries $V \in U(d^n)$**
 - Requires up to $\lceil n/\log_2 d \rceil$ ancilla

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Sard's Theorem

Def: A **critical value** of a smooth function of smooth manifolds $f : M \rightarrow N$ is any $n \in N$ such that there is some $p \in M$ with $f(p) = n$ with the linear map of tangent spaces $(df)_p : T_p M \rightarrow T_n N$ not onto.

Sard's theorem: The set of critical values of any smooth map has measure zero.

Corollary: If $\dim M < \dim N$, then **image(f)** is measure 0.

- $U(\ell)$ is a smooth submanifold of $\mathbb{C}^{\ell \times \ell}$, (real) dimension ℓ^2
- Qudit circuit topology τ with k two-qudit gates $V \in U(d^2)$ induces smooth evaluation map $f_\tau : U(1) \times U(d^2)^k \rightarrow U(d^n)$

Dimension-Based Bounds

- Simple gate library: $V \in U(d^2)$, two-qudit unitaries
- Consequence: Any universal circuit must contain **at least d^{2n}/d^4 two-qudit gates**
 - Finer analysis often possible
 - (Countably) Many circuits with too few boxes will not help
- **$h(n) \in \Omega[f(n)]$** means $h(n) \geq C f(n)$, some C and all $n \geq 1$
- Universality requires $\Omega(d^{2n})$ two-qudit gates

State-Synthesis Lower Bounds

- **State-Synthesis Problem:** Given $|\psi\rangle \in \mathcal{H}(n, d)$, construct small circuit realizing unitary U such that $U|0\rangle = |\psi\rangle$
- Projection of matrix on first column: **smooth map**
- $\dim_{\mathbb{R}} \mathcal{H}(n, d) = 2d^n$
- **Result:** At least $2d^n/d^4 \in \Omega(d^n)$ two-qudit gates for state synthesis

More Notation for Asymptotics

- $h(n) \in O[f(n)]$ means $h(n) \leq Cf(n)$, all $n \geq 1$
- $h(n) \in \Theta[f(n)]$ means both $h(n) \in \Omega[f(n)]$, $h(n) \in O[f(n)]$
- If a circuit problem requires $\Theta[f(n)]$ gates, i.e. $C_1f(n)$ gates required & $C_2f(n)$ construction exists ($C_1 < C_2$), then a $O[f(n)]$ gate construction is **asymptotically optimal**
- Construction of $V \in U(d^n)$ of last section needs $O(n^2d^{2n})$ gates, **not asymptotically optimal**

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Notation: $\bigwedge(C, V)$

- $V \in U(d)$, $C = [C_1 C_2 \dots C_n]$ length n word from $\{0, 1, \dots, d-1\} \sqcup \{*\} \sqcup \{T\}$
Only one T allowed

- $\bigwedge(C, V)$: n -qudit operator, applies V to the qudit of T iff all $\#$'s match

- Case $C_n = T$:

$$\bigwedge([C_1 C_2 \dots C_{n-1} T], V) |c_1 c_2 \dots c_n\rangle = \begin{cases} |c_1 \dots c_{n-1}\rangle \otimes V |c_n\rangle, & c_j = C_j \text{ or } C_j = * \\ |c_1 \dots c_{n-1} c_n\rangle, & \text{else} \end{cases}$$

- Case $C_j = T$, $j < n$: χ_j^n SWAP by $|j\rangle \leftrightarrow |n\rangle$, $\tilde{C} = [C_1 C_2 \dots C_{j-1} C_n C_{j+1} \dots C_{n-1} T]$

$$\bigwedge(C, V) = \chi_j^n \bigwedge(\tilde{C}, V) \chi_j^n$$

State-Synthesis Notation

- State-Synthesis Equation: $\prod_{k=1}^p \Lambda[C(k), V(k)^\dagger] |0\rangle = |\psi\rangle$
- **Adjoint Householder**: $\prod_{k=1}^p \Lambda[C(p-k+1), V(p-k+1)] |\psi\rangle = |0\rangle$
- One-qudit Answer: **Householder** W , $W|\psi\rangle = |0\rangle$

$$\begin{cases} |\eta\rangle &= |\psi\rangle - \sqrt{\langle\psi|\psi\rangle} \frac{\langle 0|\psi\rangle}{|\langle 0|\psi\rangle|} |0\rangle \\ W &= I_d - (2/\langle\eta|\eta\rangle) |\eta\rangle\langle\eta| \end{cases}$$
- **Asymptotically optimal**: In n -qudits, solve so total # of control boxes $\leq Cd^n$ (See slide 16.)

Clubsuit Householder Reduction

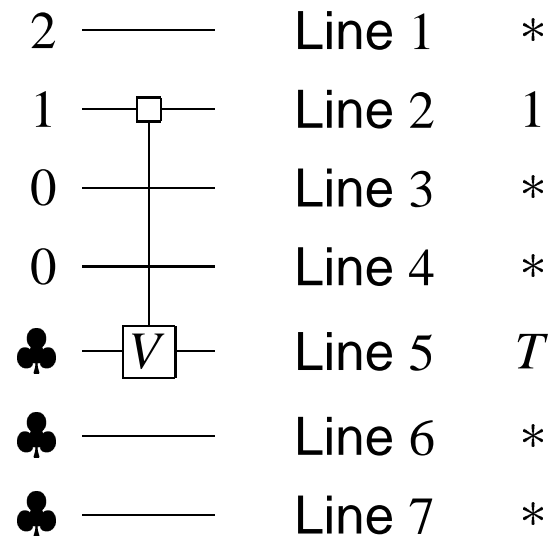
- **Our Answer:** $p = (d^n - 1)/(d - 1)$, one control only each $\mathcal{C}(k)$
- **Circuit:** Topology recursive, $V(k)$ are not
- Topology built using **♣-sequence**
- Terms $\{s_j\}_{j=1}^{(d^n-1)/(d-1)}$: **word from alphabet $\{0, 1, \dots, d-1\} \sqcup \{\clubsuit\}$**
 - #’s encode amplitudes to zero
 - ♣ is stop-symbol, places T in $\mathcal{C}(k)$

Clubsuit Householder Reduction Cont.

- Constructing n^{th} ♣-sequence
 - In order, write dit s to sequence
for every lead dit and every s in $(n-1)^{\text{st}}$ sequence
 - Append n -fold ♣♣...♣
- Qutrit ($d = 3$) sample:

n	♣-sequence, $d = 3$
1	♣
2	0♣, 1♣, 2♣, ♣♣
3	00♣, 01♣, 02♣, 0♣♣, 10♣, 11♣, 12♣, 1♣♣, 20♣, 21♣, 22♣, 2♣♣, ♣♣♣
4	000♣, 001♣, 002♣, 00♣♣, 010♣, 011♣, 012♣, 01♣♣, 020♣, 021♣, 022♣, 02♣♣, 0♣♣♣ 100♣, 101♣, 102♣, 10♣♣, 110♣, 111♣, 112♣, 11♣♣, 120♣, 121♣, 122♣, 12♣♣, 1♣♣♣ 200♣, 201♣, 202♣, 20♣♣, 210♣, 211♣, 212♣, 21♣♣, 220♣, 221♣, 222♣, 22♣♣, 2♣♣♣, ♣♣♣♣

Clubsuit Householder Reduction Cont.

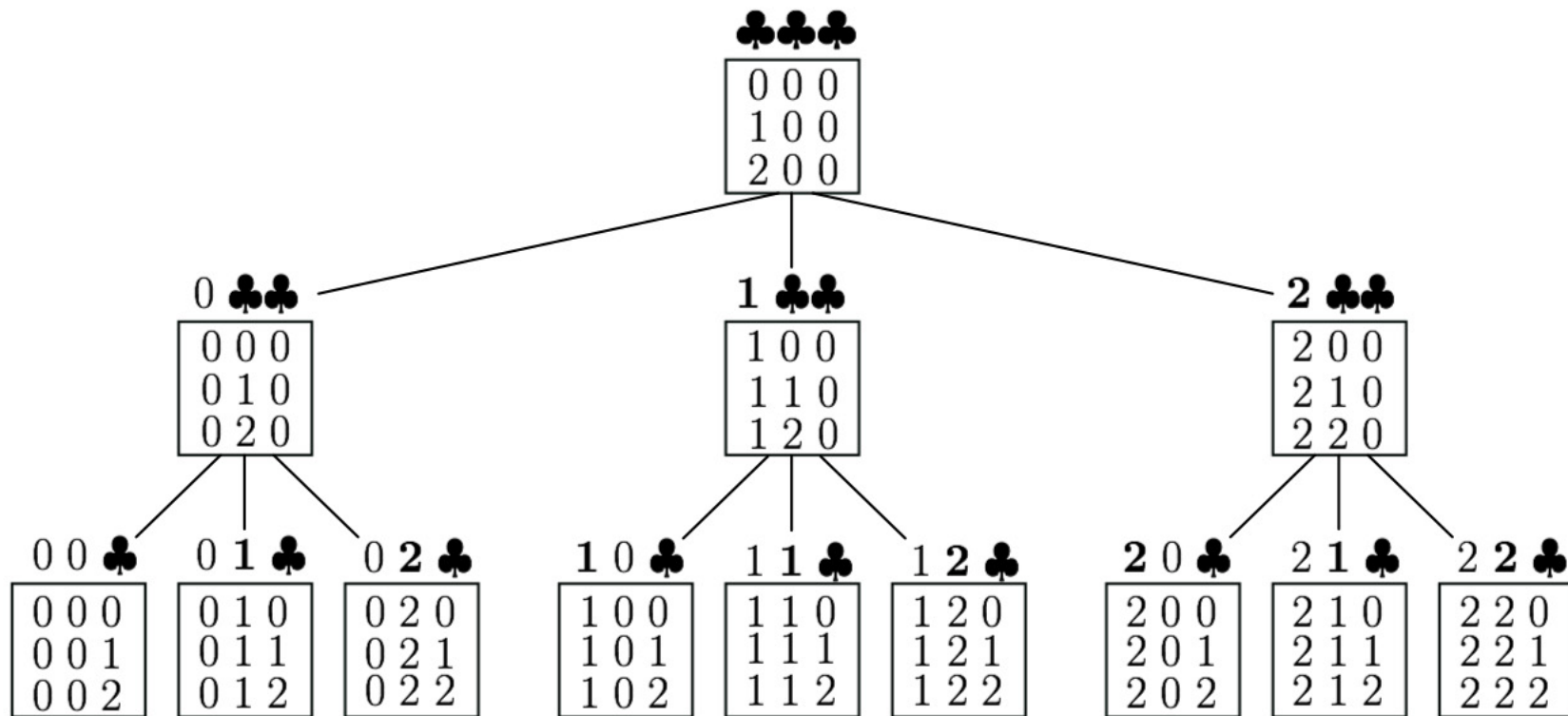


- $\Lambda(C, V)$ given V , term $t = 2100♣♣♣♣$, **one** control, first $\neq 0$ over first $♣$
- T at top $♣$, V a $d \times d$ Householder to **zero amplitudes** $2100\ell 00$, $\ell \geq 1$

Clubsuit Householder Reduction Summary

- **Result:** $\prod_{k=1}^p \Lambda[C(p-k+1), V(p-k+1)] |\psi\rangle = |0\rangle$
 - $p = (d^n - 1)/(d - 1) \in O(d^n)$
 - Each $C(k)$, at most **one control** \implies two-qudit gate
 - **Delicate argument (omitted):** no $\Lambda[C(k), V(k)]$ destroys 0-amplitudes created by $\Lambda[C(j), V(j)]$, $j < k$
- **Result:** Asymptotically Optimal State-Synthesis

Zeroes Introduced for 3 Qutrits: Lower 2 Entries from Depth-First Search of Tree



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Householder $U|\psi\rangle = |j\rangle, j \neq 0$

- $\Lambda(\tilde{C}, \tilde{V})$: Similary transform of $\Lambda(C, V)$ by $\otimes_{\ell=1}^n \oplus d_\ell$

$$\begin{aligned}\tilde{C}_k &= \begin{cases} *, & C_k = * \\ T, & C_k = T \\ (C_k + d_k) \bmod d, & C_k \in \{0, 1, \dots, d-1\} \end{cases} \\ \tilde{V} &= (\oplus d_q) V (\oplus d - d_q), \text{ where } C_q = T\end{aligned}$$

- **Notice:** $|j\rangle = [\otimes_{\ell=1}^n (\oplus d_\ell)] |0\rangle$ for $j = d_1 \dots d_n$
 - Compute $|\phi\rangle = [\otimes_{\ell=1}^n \oplus d - d_\ell] |\psi\rangle$
 - Produce $\prod_{k=1}^p \Lambda[C(p-k+1), V(p-k+1)] |\phi\rangle = |00 \dots 0\rangle$
 - **Consequence:** $\prod_{k=1}^p \Lambda[\tilde{C}(p-k+1), \tilde{V}(p-k+1)] |\psi\rangle = |j\rangle$

Triangle Algorithm: Optimal n -qudit Unitary Circuit

- for each of d block columns, width d^{n-1}
 - Triangulate $d^{n-1} \times d^{n-1}$ diagonal subblock, **recursive**, extra control
 - for each of d^{n-1} columns $j = d_{n-1}d_{n-2} \dots d_1$ in block-column
 - * Use ♣-Householder to zero each subcol below block diagonal onto $d_{n-1}d_{n-2} \dots d_1$ -amplitude
 - * Use $\wedge([Td_{n-1}d_{n-2} \dots d_1], V)$ to clear remaining subdiagonal entries
- **Makes more sense after watching movie**

Number of Control Boxes for Triangle

- $O(d^{2n})$ control boxes: **Asymptotically Optimal # of Gates from $U(d^2)$**

n	d	2	3	4	5	6	7	8
2		5	17	39	74	125	195	287
3		40	285	1 140	3 370	8 820	17 535	33 880
4		220	3 240	22 176	100 000	345 060	987 840	2 464 000
5		1 040	32 130	379 776	2 631 500	12 931 920	49 999 110	
6		4 560	301 239	6 220 032	66 768 750	470 221 200		
7		19 200	2 757 807	100 279 728	1 676 043 750			
8		79 040	24 994 494	1 608 794 112				
9		321 280	225 584 676					
10		1 296 640	2 032 525 629					
11		5 212 160	1 120 813 409					
12		20 904 960						

Conclusions

- Qudits are much like qubits
 - Qubits: Optimal # of two-qubit gates is $\Theta(4^n)$
 - Clever constructions: $\Theta(d^{2n})$, not $O(n^2 d^{2n})$ and $\Omega(d^{2n})$
- Tensor structures are not fungible
- Qudit emulation with qubits is bad design
- Complexities $\Theta(d^{2n})$, dits vs $\Theta(2^{2n})$, bits: asymptotically dits are no better & no worse

<http://www.arxiv.org> **Coordinates**

- *QR* techniques, first use of CINC: [q-ph/0002033](#)
- Qudit Fourier transforms: [q-ph/9702028](#)
- Restricted one-qudit universality: [q-ph/0407223](#)
- Asymptotically Optimal n -qudit universality: [q-ph/0410116](#)